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论文题目：半单模的 GM 相关条件

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半单模的GM相关条件

摘要: 设 R 是一个环。如果对于任意的 $x \in R$，都存在 R 中的一个可逆元 u 使得 $x+u$ (或 $x-u$) 和 $x-u^{-1}$ 是 R 中的可逆元，则称 R 满足 (P) (或 (Q))条件。这两个条件和 GM条件有着密切的联系，并且是GM条件的推广。 GM条件最早在[5]中在作者研究满足 单位1稳定条件的环时被加以讨论，并由Chen[1] 在2010年命名为Goodearl-Menal条件。这里我们简称其为GM条件。 在[6]中，作者证明了如果 D 是满足 $|D|>3$ 的除环，则 $End(V_D)$ 满足 (P) 条件；如果 D 是满足 $|D|>2$ 的除环，则 $End(V_D)$ 满足 (Q) 条件，其中 V_D 是 D 上任意维的右向量空间。这样，这个重要的环类满足 (P) (或 (Q)) 条件。 在本文中，我们证明了在一定条件下，对于一个任意维数的半单模 M_R，$End(M_R)$ 也是满足 (P) (或 (Q)) 条件的。

关键词: 半单环，(P) 条件，(Q)条件，Goodearl-Menal 条件

GM Related Conditions for Semisimple Modules

Abstract: A ring R is said to satisfy condition (P) (resp. (Q)) if for any $x \in R$, there is a unit u of R such that both $x+u$ (resp. $x-u$) and $x-u^{-1}$ are units of R, which are closely related to and are generalizations of the Goodearl-Menal condition came up by Chen[1] in 2010. For brevity, we call it the GM condition. It is proved in [6] that $End(V_D)$ satisfies (P) if $|D|>3$, and satisfies (Q) if $|D|>2$, where V_D is right vector space over a division ring D with arbitrary dimension. Thus, these important class of rings satisfy condition (P) (resp. (Q)). Here we proved that for a right semisimple module M_R with arbitrary uniform dimension, $End(M_R)$ satisfies conditions (P) (resp. (Q)) under certain restrictions.

Keywords: semisimple ring, condition (P), condition (Q), Goodearl-Menal condition

1. Introduction

Following Menal and Moncasi [5], a ring R is said to satisfy unit 1-stable range if whenever $aR + bR = R$, there exists $u \in U(R)$ such that $a + bu \in U(R)$. The unit 1-stable range is always satisfied by a ring R that for any $x, y \in R$, there is a unit u of R such that both $x + u$ and $y - u^{-1}$ are units of R (see[2]). The latter condition is called the Goodearl-Menal condition by Chen [1], and has been discussed by several authors. We write it as the GM condition for brevity. Clearly, the GM condition implies condition (Q). If in addition R is a semilocal ring, it also implies condition (P). In [7], Zelinsky proved that every element in $End(V_D)$ is a sum of two units where V_D is an arbitrary dimensional right vector space over a division ring D unless $D = \mathbb{Z}_2$ and $dimV = 1$. Motivated by this, Chen[1] proved that $End(V_D)$ satisfies (P) if $|D| \neq 2, 3$, and satisfies (Q) if $|D| \neq 2$, where V_D is an countably generated right vector space over a division ring D. Later in [6], the authors generalized the results and proved that $End(V_D)$ satisfies (P) if $|D| > 3$, and satisfies (Q) if $|D| > 2$, where V_D is an arbitrary dimensional right vector space over a division ring D. Here we proved similar results for a semisimple right R module M_R, which are generalizations of the above results.

Throughout R is an associative ring with identity. We write $U(R)$ for the set of units of R. For a right R module M_R, the ring of endomorphisms of M_R is denoted by $End_R(M_R)$ or $End(M_R)$ for convenience. Write $J = J(R)$ for the Jacobson radical of a ring R. If X is a set, we denote $|X|$ as the cardinal of X.

2. Main Results

Definition 2.1. *$M_R = \oplus_{i=1}^{\infty} M_i$, $M_i \cong M_j \neq 0$, $\forall\ i, j = 1, 2, \cdots$, are simple isomorphic R-modules. $f \in End(M_R)$ is called a shift operator if there exist $0 \neq v_i \in M_i$, $i = 1, 2, \cdots$, such that $f(v_i) = v_{i+1}$ for all $i = 1, 2, \cdots$.*

Definition 2.2. [6] *A ring R satisfies condition (P) if for any $a \in R$, there exists some $u \in U(R)$ such that $a+u, a-u^{-1} \in U(R)$; satisfies condition (Q) if for any $a \in R$, there exists some $u \in U(R)$ such that $a-u, a-u^{-1} \in U(R)$.*

Definition 2.3. [1] *A ring R satisfies the Goodearl-Menal (or GM)condition if for any $x, y \in R$, there is a unit u of R such that both $x + u$ and $y - u^{-1}$ are units of R.*

Definition 2.4. [3] *For a right R module M_R, we say M_R has uniform dimension n (write $u.dimM = n$) if n is the supremum of the set $\{k :$ M contains a direct sum of k nonzero submodules$\}$.*

Lemma 2.5. *$\mathbb{M}_n(D)$ where $n \geq 1$, D is a division ring with $|D| \geq 4$, $\mathbb{M}_n(\mathbb{Z}_3)$ where $n \geq 2$ and $\mathbb{M}_n(\mathbb{Z}_2)$ where $n \geq 3$ satisfy Goodearl-Menal condition. In particular, these rings satisfy conditions (P) and (Q).*

Proof. See [4, Lemma 2.2, Lemma 2.7, Lemma 2.8]. □

Lemma 2.6. *Let L be a finite index set, $|L| = n \geq 1$, $M_R = \oplus_{i\in L} M_i$. $M_i \cong M_j \neq 0$, $i, j \in L$ are simple R-modules, and $D_i := End(M_i) \cong D$ for all $i \in L$ and a division ring D.*

(1) *If $|D| > 3$, then $End(M_R)$ satisfies (P).*
(2) *If $|D| > 2$, then $End(M_R)$ satisfies (Q).*

Proof. (1). $End(M_R) \cong \mathbb{M}_n(D)$, and $\mathbb{M}_n(D)$ satisfies (P) for $|D| > 3$ by Lemma 2.5, so (1) holds.

(2). Similar to the proof of (1). □

Lemma 2.7. *$M_R = \oplus_{i=1}^{n} M_i$, $M_i \cong M_j \neq 0$, $i, j = 1, 2, \cdots$, are simple R-modules. If $f \in End(M_R)$ is a shift operator, then there exists a unit $g \in End(M_R)$ such that $f+g, f-g, f-g^{-1}$ are units of $End(M_R)$.*

Proof. Assume $0 \neq v_i \in M_i$, $i = 1, 2, \cdots, n$, such that $f(v_i) = v_{i+1}$ for all $i = 1, 2, \cdots, n$. Let $g \in End(M_R)$ be given by $v_{2n-1} \mapsto v_{2n-1} + v_{2n}$ and $v_{2n} \mapsto v_{2n}$ for $n \geq 1$. Then g is well defined and $g \in U(End(M_R))$ with g^{-1} given by $v_{2n-1} \mapsto v_{2n-1} - v_{2n}$ and $v_{2n} \mapsto v_{2n}$ for $n \geq 1$. Thus, it is easy to get the following facts:

$f-g \in U(End(M_R))$ is given by $v_{2n-1} \mapsto -v_{2n-1}$ and $v_{2n} \mapsto -v_{2n}+v_{2n+1}$ for $n \geq 1$ with $(f-g)^{-1}$ given by $v_{2n-1} \mapsto -v_{2n-1}$ and $v_{2n} \mapsto -v_{2n} - v_{2n+1}$ for $n \geq 1$;

$f - g^{-1} \in U(End(M_R))$ is given by $v_{2n-1} \mapsto -v_{2n-1} + 2v_{2n}$ and $v_{2n} \mapsto -v_{2n} + v_{2n+1}$ for $n \geq 1$ with $(f-g^{-1})^{-1}$ given by $v_{2n-1} \mapsto -v_{2n-1} - 2v_{2n} - 2v_{2n+1} - 4v_{2n+2}$ and $v_{2n} \mapsto -v_{2n} - v_{2n+1} - 2v_{2n+2}$ for $n \geq 1$;

$f + g \in U(End(M_R))$ is given by $v_{2n-1} \mapsto v_{2n-1} + 2v_{2n}$ and $v_{2n} \mapsto v_{2n} + v_{2n+1}$ for $n \geq 1$ with $(f+g)^{-1}$ given by $v_{2n-1} \mapsto v_{2n-1} - 2v_{2n} + 2v_{2n+1} - 4v_{2n+2}$ and $v_{2n} \mapsto v_{2n} - v_{2n+1} + 2v_{2n+2}$ for $n \geq 1$. □

Theorem 2.8. *Let I be an arbitrary index set, and $M_R = \oplus_{i\in I} M_i$, $M_i \cong M_j \neq 0$, $i, j \in I$ are simple R-modules. Let $D_i := End(M_i) \cong D$ for all $i \in I$ and a division ring D.*

(1) *If $|D| > 3$, then $End(M_R)$ satisfies (P).*
(2) *If $|D| > 2$, then $End(M_R)$ satisfies (Q).*

Proof. (1) Let $f \in End(M_R)$. Denote $S = \{(W, g) | f(W) \subseteq W,\ W \leq M_R,\ and\ g,\ f|_W + g,\ f|_W - g^{-1} \in U(End(M_R))\}$. Then it is easy to see that $((0), id) \in S$, where id is the identity isomorphism of M_R. If we define $(U_1, g_1) < (U_1, g_1)$ for two elements (U_1, g_1), $(U_2, g_2) \in S$ by $U_1 \subseteq U_2$ and $g_2|_{U_1} = g_1$, then S is an inductive set. So by Zorn's Lemma, there exists a maximal element of S, say, (W, g). Then it suffices to show that $W = M_R$.

Suppose $W \subsetneq M_R$. Then there exists some $0 \neq x \in M \backslash W$. Define $W_0 := W + K$, where $K = span\{x,\ f(x),\ f^2(x),\ f^3(x),\ \cdots\}$. Then $K_R \leq M_R$ is a f-subspace. Since M_R is semisimple, we can write $W_0 = W \oplus N$ for some $0 \neq N_R \leq M_R$. Define the following homomorphisms:

$\overline{f} : W_0/W \to W_0/W$ defined by $\overline{f}(\overline{w_0}) = \overline{f(w_0)}$, $w_0 \in W_0$;

$\pi : W_0 \to N$ is the obvious epimorphism;

$\varphi : W_0/W \to N$ is the isomorphism defined by $\varphi(\overline{w_0}) = \pi(w_0)$ for all $w_0 \in W_0$;

$\theta := \varphi\overline{f}\varphi^{-1} : N \to N$. Then $\theta\varphi = \varphi\overline{f}$.

If follows that $V/W_0 = span\{\overline{x}, \overline{f}(\overline{x}), \overline{f}^2(\overline{x}), \cdots\}$. So $N = span\{\varphi(\overline{x}), \varphi(\overline{f}(\overline{x})), \varphi(\overline{f}^2(\overline{x})), \cdots\} = span\{\varphi(\overline{x}), \theta\varphi(\overline{x}), \theta^2\varphi(\overline{x})), \cdots\}$. If $u.dimN = n < \infty$ for some $n \geq 1$, then there exists a subset $K \subseteq I$, $|K| = n$, such that $N = \oplus_{k\in K} M_k$. By Lemma 2.6, $\theta \in End(N)$ satisfies (P). If $u.dimN = \infty$, then $\theta \in End(N)$ is a shift operator. By Lemma 2.7, θ also satisfies (P). This implies that there exists some $\alpha \in U(End(N))$ such that $\theta+\alpha$, $\theta-\alpha^{-1}$ are units of $End(N)$.

Let $h : W_0 \to W_0$ be given by $h(w+z) = g(w) + \alpha(z)$, where $w \in W$, $z \in N$. Then it is easy to verify that $h \in U(End(W_0))$. We next show that $f+h, f-h^{-1} \in U(End(W_0))$.

Let $w \in W$, $z \in N$.

$(*)$ $(f+h)(w+z) = (f+g)(w) + (f(z) + \alpha(z))$

$(**)$ $\pi(f+h)(w+z) = \pi f(z) + \alpha(z) = \varphi(\overline{f}(\overline{z})) + \alpha(z) = \theta\varphi(\overline{z}) + \alpha(z) = \theta\pi(z) + \alpha(z) = \theta(z) + \alpha(z) = (\theta+\alpha)(z)$

If $(f+h)(w+z) = 0$, then by $(**)$, $(\theta+\alpha)(z) = 0$. But $\theta+\alpha \in U(End(N))$, so $z = 0$. This implies that $(f+g)(w) = 0$ by $(*)$ and so $w = 0$ since $f+g \in U(End(W))$. Accordingly, $f+g$ is a monomorphism.

From $(*)$, $W \subseteq Im(f+h)$. $\forall z \in N$, there exists $y \in N$ such that $(\theta+\alpha)(y) = z$ since $\theta+\alpha \in U(End(N))$. So according to $(**)$, we get $z = (\theta+\alpha)(y) = \pi(f+h)(w+z) \subseteq Im(f+h)$ since $W \subseteq Im(f+h)$. Thus, $f+g$ is an epimorphism.

Therefore, $(W_0, h) \in S$ and $(W, g) < (W_0, h)$, which is contradict to the assumption. Hence, $W = M_R$ and we complete the proof.

(2). Similar to the proof of (1). □

Corollary 2.9. *M_R is semisimple.*

1. *If $|End(N)| > 3$ for any simple submodule N of M, then $End(M_R)$ satisfies (P).*
2. *If $|End(N)| > 2$ for any simple submodule N of M, then $End(M_R)$ satisfies (Q).*

Proof. (1). Write $M_R = \oplus_{j\in J} M_i$, $N_j = \oplus_{i_j\in I_j} M_{j_i}$, where J and $\{I_j\}_{j\in J}$ are index set, satisfying $M_{j_i} \cong M_{j_k}$ for $j_i, j_k \in I_j$. By Theorem 2.8, $End(N_j)$, $j \in J$ satisfies (P). Thus, $End(M_R) \cong \Pi_{j\in J} End(N_j)$ also satisfies (P).

(2). Similar to the proof of (1). □

Corollary 2.10. *Let R be a semisimple ring such that $R/J(R)$ is semisimple and P_R is projective with $PJ \ll P$.*

1. *If $|End(N)| > 3$ for any simple submodule N of $(P/PJ)_{R/J(R)}$, then $End_R(P_R)$ satisfies (P).*
2. *If $|End(N)| > 2$ for any simple submodule N of $(P/PJ)_{R/J(R)}$, then $End_R(P_R)$ satisfies (Q).*

Proof. (1). $End_R(P_R)/J(End_R(P_R)) \cong End_R(P_R) \cong End_{R/J}(P_R)$. But $End_{R/J}(P_R)$ satisfies (P) by Corollary 2.9, so does $End_R(P_R)/J(End_R(P_R))$. Thus, $End_R(P_R)$ satisfies (P) since units can be be lifted modulo $J(End_R(P_R))$.

(2). Similar to the proof of (1). □

Using Corollary2.10, we can deduce the following results immediately:

Corollary 2.11. *Let R be a right perfect ring and P_R be a projective module. Then*

(1) *If $|End(N)| > 3$ for any simple submodule N of $(P/PJ)_{R/J(R)}$, then $End_R(P_R)$ satisfies (P).*

(2) *If $|End(N)| > 2$ for any simple submodule N of $(P/PJ)_{R/J(R)}$, then $End_R(P_R)$ satisfies (Q).*

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